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Joint Spectra and Nilpotent Lie Algebras of Linear Transformations

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ABSTRACT

Given a complex nilpotent finite dimensional Lie algebra of linear transformations, L, in a complex finite dimensional vector space, E, we study the joint spectra $\mathrm{Sp}(L, E)$, $\sigma_{\delta,k}(L, E)$, and $\sigma_{\pi,k}(L, E)$. We compute them, and we prove that they all coincide with the set of weights of L for E. We also give a new interpretation of some basic module operations of the Lie algebra L in terms of the joint spectra. © 1997 Elsevier Science Inc.

1. INTRODUCTION

The Taylor joint spectrum is one of the most dynamic and powerful subjects of multiparameter spectral theory. This spectrum, which was introduced in 1970 by J. L. Taylor, [9], associates to an n-tuple $a = (a_1, \dots, a_n)$ of mutually commuting bounded linear operators on a Banach space E—i.e., $a_i \in \mathcal{L}(E)$, the algebra of all bounded linear operators on E, and $[a_i, a_j] = 0 \, \forall i, j, 1 \leqslant i, j \leqslant n$—a compact nonempty subset of $\mathbb{C}^n$, which we denote by $\mathrm{Sp}(a, E)$. When $n = 1$, the Taylor joint spectrum reduces to the classical spectrum of a single operator, and if $\tilde{a} = (a_i, \dots, a_k)$ is the k-tuple, $1 \leqslant k \leqslant$

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$n-1$, of the first k operators of a, then the Taylor joint spectrum satisfies the so-called projection property, $\mathrm{Sp}(\tilde{a}, E) = \pi\,\mathrm{Sp}(a, E)$, where π is the projection map of $\mathbb{C}^n$ onto $\mathbb{C}^k$.

In addition, if E is a complex finite dimensional vector space, M. Chò and M. Takaguchi compute in [4] the Taylor joint spectrum of an n-tuple a of commuting operators on E, and they obtain that $\mathrm{Sp}(a, E) = \sigma_{\mathrm{pt}}(a)$, the joint point spectrum of a, where $\sigma_{\mathrm{pt}}(a) = \{\lambda \in \mathbb{C}^n, \exists x \in E,\ x \neq 0 : a_j(x) = \lambda_j x,\ 1 \leqslant j \leqslant n\}$. Furthermore, in [6], A. McIntosh, A. Pryde, and W. Ricker, as a consequence of a more general result which concerns infinite dimensional spaces too, also compute the Taylor joint spectrum for an n-tuple of linear transformations on a complex finite dimensional vector space, and they compare it with other joint spectra.

In [2] we defined a joint spectrum for complex solvable finite dimensional Lie algebras of operators, L, acting on a Banach space E, and we denoted it by $\mathrm{Sp}(L, E)$. We also proved that $\mathrm{Sp}(L, E)$ is a compact nonempty subset of L^*, and that the projection property for ideals still holds. When L is a commutative algebra, $\mathrm{Sp}(L, E)$ reduces to the Taylor joint spectrum in the following sense. If $\dim L = n$ and if $\{a_i\}_{(1 \leqslant i \leqslant n)}$ is a basis of L, we consider the n-tuple $a = (a_1, \ldots, a_n)$; then $\{(f(a_1), \ldots, f(a_n));\ f \in \mathrm{Sp}(L, E)\} = \mathrm{Sp}(a, E)$, i.e., $\mathrm{Sp}(L, E)$, in terms of the basis of L^* dual of $\{a_i\}_{(1 \leqslant i \leqslant n)}$, coincides with the Taylor joint spectrum of the n-tuple a. Then the following question arises naturally. If L is a complex Lie algebra of linear transformations in a complex finite dimensional vector space, what can be said about its joint spectrum? In this article we compute, under the above assumptions, the joint spectrum of a nilpotent Lie algebra; moreover, we extend to the nilpotent case the characterizations of [4] and [6]. In addition, our result has a beautiful feature: $\mathrm{Sp}(L, E)$ is the set of all weights of the Lie algebra L for the vector space E.

However, we consider a more general situation. In [1] we introduced a family of joint spectra, for complex solvable finite dimensional Lie algebras of operators acting on a Banach space, which, when the algebra is a commutative one, reduce to the Slodkowski joint spectra in the same sense that we have explained for the Taylor joint spectrum; see [8]. We also proved the most important spectral properties for these joint spectra: compactness, nonemptiness, and the projection property. In this article we also compute these joint spectra for complex nilpotent Lie algebras of linear transformations in a complex finite dimensional vector spaces. Moreover, we also prove that all the joint spectra considered in this article coincide with the set of weights of the Lie algebra L for the vector space E.

We observe that, though all joint spectra introduced in [1] and in [2] were defined for complex solvable Lie algebras, we restrict ourselves to the nilpotent case; for in the solvable nonnilpotent case our result fails.

The paper is organized as follows. In Section 2 we review several definitions and results of [1] and [2]. We also review several facts related to the theory of complex nilpotent Lie algebras of linear transformations in complex finite dimensional vector spaces. In Section 3 we prove our main theorems and, with them, we give a proof of the compactness and nonemptiness of the joint spectra, for the case under consideration, different from that in [1] and [2]. We also show an example of a solvable nonnilpotent Lie algebra of linear transformations where our result fails. In Section 4 we consider some basic module operations of the Lie algebra L in order to compute several joint spectra.

2. PRELIMINARIES

We briefly recall several definitions and results related to the spectrum of a Lie algebra (see [2]). In [2] we considered complex solvable finite dimensional Lie algebras of linear bounded operators acting on a Banach space; however, for our purpose, in this article we restrict ourselves to the case of complex finite dimensional nilpotent Lie algebras of linear transformations defined on finite dimensional vector spaces.

From now on, E denotes a complex finite dimensional vector space, $\mathscr{L}(E)$ the algebra of all linear transformations defined on E, and L a complex nilpotent finite dimensional Lie subalgebra of $\mathscr{L}(E)^{\mathrm{op}}$, i.e., the algebra $\mathscr{L}(E)$ with its opposite product. Such an algebra is called a nilpotent Lie algebra of linear transformations in the vector space E. If $\dim L = n$ and f is a character of L, i.e., $f \in L^*$ and $f(L^2) = 0$, where $L^2 = \{[x, y]; x, y \in L\}$, then let us consider the chain complex $(E \otimes \wedge L, d(f))$, where $\wedge L$ denotes the exterior algebra of L, and $d_p(f)$ is as follows:

$$d_p(f) : E \otimes \wedge^p L \to E \otimes \wedge^{p-1} L,$$

$$\begin{aligned} &d_p(f)e\langle x_1 \wedge \cdots \wedge x_p\rangle \\ &\quad = \sum_{k=1}^{p} (-1)^{k+1} e[x_k - f(x_k)]\langle x_1 \wedge \cdots \wedge \hat{x}_k \wedge \cdots \wedge x_p\rangle \\ &\qquad + \sum_{1 \leqslant k < l \leqslant p} (-1)^{k+l} e\langle [x_k, x_l] \wedge x_1 \cdots \hat{x}_k \cdots \hat{x}_1 \cdots x_p\rangle, \end{aligned}$$

where ˆ means deletion. If $p \leqslant 0$ or $p \geqslant n + 1$, we define $d_p(f) = 0$.

If we denote by $H_*((E \otimes \wedge L, d(f))$ the homology of the complex $(E \otimes \wedge L, d(f))$, we may state out first definition.

DEFINITION 1. With E, L, f as above, the set $\{f \in L^*;\ f(L^2) = 0, H_*((E \otimes \wedge L, d(f))) \neq 0\}$ is the joint spectrum of L acting on E, and it is denoted by $\mathrm{Sp}(L, E)$.

As we have said, in [2] we proved that $\mathrm{Sp}(L, E)$ is a compact nonempty subset of L^*, which reduces to the Taylor joint spectrum when L is a commutative algebra, in the sense explained in the Introduction. If I is an ideal of L, and π denotes the projection map from L^* to I^*, then

$$\mathrm{Sp}(I, E) = \pi(\mathrm{Sp}(L, E)),$$

i.e., the projection property for ideals still holds. With regard to this property, we ought to mention the paper of C. Ott [7], who pointed out a gap in the proof of this result, and gave another proof of it. In any case, the projection property remains true.

We now give the definition of the Slodkowski joint spectra for Lie algebras. As in [1], we give a homological version. However, as we deal with linear transformations defined on finite dimensional vector spaces, we slightly modify our definition in order to adapt it to our case; for a complete exposition of the subject see [8] and [1].

If L and E are as above, let us consider the set

$$\Sigma_p(L, E) = \{f \in L; f(L^2) = 0, H_p((E \otimes \wedge L, d(f))) \neq 0\},$$

where $0 \leqslant p \leqslant n$ and $\dim L = n$. We now state our definition of the Slodkowski joint spectra for Lie algebras.

DEFINITION 2. With L and E as above, $\sigma_{\delta, k}$ and $\sigma_{\pi, k}$ are the following joint spectra:

$$\sigma_{\delta, k}(L, E) = \bigcup_{0 \leqslant p \leqslant k} \Sigma_p(L, E), \qquad \sigma_{\pi, k}(L, E) = \bigcup_{k \leqslant p \leqslant n} \Sigma_p(L, E),$$

where $0 \leqslant k \leqslant n$ and $n = \dim L$.

We observe that $\sigma_{\delta, n}(L, E) = \sigma_{\pi, 0}(L, E) = \mathrm{Sp}(L, E)$. In [1] we showed that $\sigma_{\delta, k}(L, E)$ and $\sigma_{\pi, k}(L, E)$, $0 \leqslant k \leqslant n$, are compact nonempty subsets of L^*, which reduce to the Slodkowski joint spectra when L is a commutative algebra. In addition, they also satisfy the projection property for ideals, i.e., if I is an ideal of L, and π denotes the projection map from L^* and I^*, then

$$\sigma_{\delta,k}(I,E) = \pi(\sigma_{\delta,k}(L,E)), \qquad \sigma_{\pi,k}(I,E) = \pi(\sigma_{\pi,k}(L,E)),$$

where k and n are as above.

We shall have occasion to use the theory of weight spaces. However, as we deal with complex nilpotent Lie algebras of linear transformation in complex finite dimensional vector spaces, we restrict our revision to the most important results of the theory, essentially Theorems 7 and 12, Chapter II, of the book by N. Jacobson, *Lie Algebras* [5]. For a complete exposition of the subject see [5, Chapter II].

Let L and E be as above. A weight of L for E is a mapping, $\alpha : x \to \alpha(x)$, of L into $\mathbb{C}$ such that there exists a nonzero vector ν in E with the property $[x - \alpha(x)]^{m_{v,x}}(v) = 0$ for all x in L and where $m_{v,x}$ belongs to $\mathbb{N}$. The set of vectors, zero included, which satisfy this condition is a subspace of E, E_α, called the weight space of E corresponding to the weight α,

$$E_\alpha = \{v \in E; \forall x \in L\ \exists m_{v,x} \in \mathbb{N} \text{ such that } [x - \alpha(x)]^{m_{v,x}}(v) = 0\}.$$

As a consequence of our assumptions we have the following properties (see [5, Chapter II, Theorems 7, 12]:

(1) The weights are linear functions on L, which vanish on L^2, i.e., they are characters of L.

(2) E has only a finite number of distinct weights; the weight spaces are submodules, and E is the direct sum of them.

(3) For each weight α, the restriction of any $x \in L$ to E_α has only one characteristic root, $\alpha(x)$, with some multiplicity.

(4) There is a basis of E such that for each weight α the matrices of elements of L in the weight space E_α can be taken simultaneously in the form

$$x_\alpha = \begin{pmatrix} \alpha(x) & * \\ 0 & \alpha(x) \end{pmatrix}.$$

Finally, if L is a complex nilpotent finite dimensional Lie algebra, by [3, IV, 1] there is a Jordan-Hölder sequence of ideals, $(L_i)_{(1 \leqslant i \leqslant n)}$, such that

(1) $\{0\} = L_0 \subseteq L_i \subseteq L_n = L$;
(2) $\dim L_i = i$;
(3) there is a $k, 0 \leqslant k \leqslant n$, such that $L_k = L^2$;
(4) if $i < j$, $[L_i, L_j] \subseteq L_{i-1}$.

As a consequence, if we consider a basis of L, $\{x_j\}_{(1 \leqslant j \leqslant n)}$, such that $\{x_j\}_{(1 \leqslant j \leqslant i)}$ is a basis of L_i, we have that

$$[x_j, x_i] = \sum_{h=1}^{i-1} c_{ij}^h x_h,$$

where $i < j$.

3. THE MAIN RESULT

In this section, we compute the joint spectra of a nilpotent Lie algebra of linear transformation in a complex finite dimensional vector space. In addition, as we have said in the introduction, under our assumptions, we give an elementary proof of the compactness and the nonemptiness of these joint spectra. We also consider an example which show that in the solvable nonnilpotent case our result fails. Let us begin with our characterization of the joint spectra.

THEOREM 1. *Let E be a complex finite dimensional vector space, and L a complex nilpotent Lie subalgebra of $\mathscr{L}(E)^{\mathrm{op}}$. Then, if α is a weight of L for E, α belongs to* $\mathrm{Sp}(L, E)$.

Proof. As E is the direct sum of its weight spaces, which are L-modules of E, by the definition of the joint spectrum and elementary homological algebra we may assume that there is only one weight of L for E. Let us denote it by α.

We now observe that, as α is a character of L, we may consider the chain complex $(E \otimes \wedge L, d(\alpha))$. Moreover , if $\{x_i\}_{(1 \leqslant i \leqslant n)}$ is the basis of L defined at the end of Section 2, it is easy to verify that

$$d_n(\alpha)e\langle x_1 \wedge \cdots \wedge x_n \rangle = \sum_{k=1}^{n} (-1)^{k+1} e[x_k - \alpha(x_k)]\langle x_1 \wedge \cdots \wedge \hat{x}_k \wedge \cdots \wedge x_n \rangle.$$

However, by the properties of the weights reviewed in Section 2, there exists an ordered basis of E, $\{e_i\}_{(1 \leq i \leq m)}$, where $m = \dim E$, such that

$$x_k - \alpha(x_k) = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix},$$

for all k, $1 \leqslant k \leqslant n$. Thus we have that

$$e_1[x_k - \alpha(x_k)] = 0.$$

Thus,

$$d_n(\alpha)e_1\langle x_1 \wedge \cdots \wedge x_n \rangle = 0,$$

which implies that α belongs to $\mathrm{Sp}(L, E)$. ∎

We now state one of our main results.

THEOREM 2. *Let E be a complex finite dimensional vector space, and L a complex nilpotent Lie subalgebra of $\mathscr{L}(E)^{\mathrm{op}}$. Then*

$$\mathrm{Sp}(L, E) = \{\alpha \in L^*;\ \alpha \textit{ is a weight of } L \textit{ for } E\}.$$

Proof. By Theorem 1, it is enough to see that,

$$\mathrm{Sp}(L, E) \subseteq \{\alpha \in L^*;\ \alpha \text{ is a weight of } L \text{ for } E\}.$$

As in Theorem 1, we may suppose that there is only one weight of L for E, which we still denote by α.

In order to conclude the proof, we consider a $\beta \in L^*$, $\beta(L^2) = 0$, such that β is not a weight of L for E—i.e., in our case, $\alpha \neq \beta$—and, by refining an argument of [2], we prove that β does not belong to $\mathrm{Sp}(L, E)$.

Let us consider the chain complex $(E \otimes \wedge L, d(\beta))$ associated to β, the sequence of ideals $(L_i)_{(1 \leqslant i \leqslant n)}$, and the basis $\{x_i\}_{(1 \leqslant i \leqslant n)}$ in L reviewed in Section 2. As $\alpha \neq \beta$, there is a j, $k + 1 \leqslant j \leqslant n$, where $k = \dim L^2$ and $n = \dim L$, such that $\alpha(x_j) \neq \beta(x_j)$. Now, if L' is the ideal of codimension 1 of L generated by $\{x_i\}_{(1 \leqslant i \leqslant n, i \neq j)}$, we have that $L' \oplus \langle x_j \rangle = L$. Moreover, we may decompose $E \otimes \wedge^p L$ in the following way:

$$E \otimes \wedge^p L = (E \otimes \wedge^p L') \oplus \left[(E \otimes \wedge^{p-1} L') \wedge \langle x_j \rangle\right].$$

If $\tilde{\beta}$ denotes the restriction of β to L', we may consider the complex $(E \otimes \wedge L', d(\tilde{\beta}))$, and, as L' is an ideal of codimension 1 of L, we may decompose $d_p(\beta)$ as follows:

$$d_p(\beta) : E \otimes \wedge^p L' \to E \otimes \wedge^{p-1} L',$$

$$d_p(\beta) = d_p(\tilde{\beta}),$$

$$d_p(\beta) : E \otimes \wedge^{p-1} L' \wedge \langle x_j \rangle \to (E \otimes \wedge^{p-1} L') \oplus \left(E \otimes \wedge^{p-2} L' \wedge \langle x_j \rangle\right),$$

$$d_p(\beta)\left(\alpha \wedge \langle x_j \rangle\right) = (-1)^p L_{p-1}(a) + \left[d_{p-1}(\tilde{\beta})(a)\right] \wedge \langle x_j \rangle,$$

where $\alpha \in E \otimes \wedge^{p-1} L'$, and L_{p-1} is the linear endomorphism defined on $E \otimes \wedge^{p-1} L'$ by

$$\begin{aligned} L_{p-1} e\langle y_1 \wedge \cdots \wedge y_{p-1} \rangle & \\ &= e\left[x_j - \beta(x_j)\right]\langle y_1 \wedge \cdots \wedge y_{p-1} \rangle \\ &\quad + \sum_{1 \leqslant l \leqslant p-1} (-1)^l e\langle [x_j, y_l] \wedge y_1 \wedge \cdots \wedge \hat{y}_1 \wedge \cdots \wedge y_{p-1} \rangle, \end{aligned}$$

where $\hat{}$ means deletion, and $\{y_h\}_{(1 \leqslant h \leqslant p-1)}$ belong to L'.

For $s \in [\![1, j-1]\!]$ we have $[x_j, x_s] = \sum_{h=1}^{s-1} c_{sj}^h x_h$, and for $s \in [\![j+1, n]\!]$ we have $[x_j, x_s] = \sum_{h=1}^{j-1} (-c_{js}^h) x_h$. Thus, by the properties of weights considered in Section 2, it is easy to see that, for each p, there is a basis of $E \otimes \wedge^p L'$ such that, in this basis, L_p is an upper triangular matrix with diagonal entries $\alpha(x_j) - \beta(x_j)$. However, as $\alpha(x_j) \neq \beta(x_j)$, L_p is an invertible map for each p. Thus, as in [2], we may construct a homotopy operator for the complex $(E \otimes \wedge L, d(\beta))$. However, this fact implies that $H_*((E \otimes \wedge L, d(\beta)) = 0$, or equivalently, β does not belong to $\mathrm{Sp}(L, E)$. ∎

As a consequence of Theorem 2, we have, under our assumptions, another proof of the compactness and nonemptiness of the spectrum.

THEOREM 3. *Let E be a complex finite dimensional vector space, and L a complex nilpotent Lie subalgebra of $\mathscr{L}(E)^{\mathrm{op}}$. Then* $\mathrm{Sp}(L, E)$ *is a compact nonempty subset of* L^*.

We now consider the joint spectra $\sigma_{\delta k}(L, E)$ and $\sigma_{\pi k}(L, E)$, $0 \leqslant k \leqslant n$. We first state a lemma which we need for our characterization of these joint spectra.

LEMMA 1. *Let E be a complex finite dimensional vector space, and L a complex nilpotent Lie subalgebra of $\mathscr{L}(E)^{\mathrm{op}}$ such that* $\dim L = n$. *Then*

(i) $\Sigma_n(L, E) = \mathrm{Sp}(L, E)$,
(ii) $\Sigma_0(L, E) = \mathrm{Sp}(L, E)$.

Proof. If one looks at Theorem 1, a consequence of its proof is that, if α is a weight of L for E, then $\alpha \in \Sigma_n(L, E)$. However, by Theorem 2 we have

$$\mathrm{Sp}(L, E) \subseteq \Sigma_n(L, E) \subseteq \mathrm{Sp}(L, E).$$

In order to prove (ii), we consider the chain complex $(E \otimes \wedge L, d(f))$ involved in the definition of $\mathrm{Sp}(L, E)$, where f is a character of L, and we study $d_1(f)$. We shall see that if α is a weight of L of E, then $d_1(\alpha)$ is not onto; equivalently, $0 \neq E/R(d_1(\alpha)) = H_0(E \otimes \wedge L, d(\alpha))$, i.e., α belongs to $\Sigma_0(L, E)$. Then, by Theorem 2, as in the proof of (i), we conclude (ii).

Let us prove the previous claim. We consider α a weight of L for E. By elementary homological arguments and the properties of weights, we may suppose that $E = E_\alpha$. If this is the case, let us consider the chain complex $(E \otimes \wedge L, d(\alpha))$ and the map $d_1(\alpha)$,

$$d_1(\alpha) : E \otimes \wedge^1 L \to E,$$

$$d_1(\alpha)(e\langle x\rangle) = e[x - \alpha(x)].$$

However, by the review in Section 2, there is a basis of E, $\{e_i\}_{(1 \leqslant i \leqslant m)}$, where $m = \dim E$, such that the matrix of any $x - \alpha(x)$, in this basis, has the form

$$x - \alpha(x) = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}.$$

Then e_m does not belong to $R(d_0(\alpha))$, which implies our claim. ■

We now may study the relation among $\mathrm{Sp}(L, E)$, $\sigma_{\delta k}(L, E)$, and $\sigma_{\pi k}(L, E)$, $0 \leqslant k \leqslant n$, for nilpotent Lie algebras of linear transformations in complex finite dimensional vector spaces. The following theorem states the relation among then.

THEOREM 4. *Let E be a complex finite dimensional vector space, and L a complex nilpotent Lie algebra of $\mathscr{L}(E)^{\mathrm{op}}$. Then we have the following identity*:

$$\mathrm{Sp}(L, E) = \sigma_{\delta k}(L, E) = \sigma_{\pi k}(L, E),$$

where $0 \leqslant k \leqslant n$; *and* $\dim L = n$.

Proof. The proof is a consequence of the following observation. By Lemma 1 and Definitions 1 and 2, we have that

$$\mathrm{Sp}(L, E) = \Sigma_0(L, E) \subseteq \sigma_{\delta k}(L, E) \subseteq \mathrm{Sp}(L, E),$$

$$\mathrm{Sp}(L, E) = \Sigma_n(L, E) \subseteq \sigma_{\pi k}(L, E) \subseteq \mathrm{Sp}(L, E),$$

where $0 \leqslant k \leqslant n$, and $\dim L = n$. ■

As a consequence of Theorems 3 and 4, we have the following result.

THEOREM 5. *Let E be a complex finite dimensional vector space, and L a complex nilpotent Lie subalgebra of $\mathscr{L}(E)^{\mathrm{op}}$. Then $\sigma_{\delta k}(L, E)$ and $\sigma_{\pi k}(L, E)$ are compact nonempty subsets of L^*, where $0 \leqslant k \leqslant n$, and $n = \dim L$.*

We now give an example in order to show that, for the solvable nonnilpotent case, our characterization of the joint spectra fails.

Let us consider the solvable Lie algebra on two generators, $L = \langle y \rangle \oplus \langle x \rangle$, with the bracket $[x, y]^{\mathrm{op}} = y$, i.e., $[x, y] = -y$. Then L may be viewed as a subalgebra of $\mathscr{L}(\mathbb{C}^2)^{\mathrm{op}}$ as follows:

$$y = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \qquad x = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}.$$

An easy calculation shows that the weights of L for $\mathbb{C}^2$ are, in terms of the dual basis of y and x, $(0, \frac{1}{2})$ and $(0, \frac{-1}{2})$. However, if we consider the previous basis, $\mathrm{Sp}(L, \mathbb{C}^2) = \{(0, \frac{1}{2}), (0, \frac{-3}{2})\}$.

4. SOME CONSEQUENCES

In this section we shall see some consequences of the main theorems. We begin with the result which relates the weights of differents ideals.

THEOREM 6. *Let E be a complex finite dimensional vector space, L a complex nilpotent Lie subalgebra of $\mathscr{L}(E)^{\mathrm{op}}$, and I an ideal of L. Then, if α is a weight of L for E, its restriction to I, $\alpha \mid I$, is a weight of I for E. Reciprocally, if β is a weight of I for E, there exists α, a weight of L for E, such that $\beta = \alpha \mid I$. In particular, if I is contained in L^2, then I has only one weight for E: $\beta = 0$.*

Proof. The proof is a consequence of Theorem 2 of Section 3 and Theorem 3 of [2], the projection property for ideals of the spectrum. ■

From now to he end of this section, we deal with two basic module operations of the Lie algebra L, the contragredient module and the tensor product; see [5, Chapter III]. We shall compute several new joint spectra by means of these module operations. Moreover, these results give a new interpretation of these module operations in terms of the joint spectra. More precisely, we consider Proposition 3 and 4 of [5, Chapter III], and we state then as properties of the joint spectra. Let us begin with the contragredient module.

If L and E are as usual, we consider the space of linear functionals on E, E^*. If x belongs to L, and if x^* denotes its adjoint, it is easy to see that the set

$$L' = \{x^*;\ x \in L\},$$

with the natural bracket, defines a complex nilpotent Lie subalgebra of $\mathscr{L}(E^*)^{\mathrm{op}}$, which is isomorphic to L with the opposite bracket. We now compute the joint spectra of L' and E^*.

THEOREM 7. *Let E be a complex finite dimensional vector space, and L a complex nilpotent Lie subalgebra of $\mathscr{L}(E)^{\mathrm{op}}$. Let E^* be the dual vector space of E, and L' the nilpotent Lie subalgebra of $\mathscr{L}(E^*)^{\mathrm{op}}$ defined above.*

Then,

$$\mathrm{Sp}(L', E^*) = \mathrm{Sp}(L, E),$$
$$\sigma_{\delta k}(L', E^*) = \sigma_{\delta k}(L, E),$$
$$\sigma_{\pi k}(L', E^*) = \sigma_{\pi k}(L, E),$$

where $0 \leqslant k \leqslant n$, *and* $n = \dim L = \dim L'$.

Proof. As we may consider the inclusion as a representation of L in $\mathscr{L}(E)^{\mathrm{op}}$, the proof is a consequence of a light modification of the proof of [5, Chapter III, 3] and Theorems 2 and 4. ■

Finally, we study the joint spectrum of a tensor product. If E_1 and E_2 are two complex finite dimensional vector spaces, and L_1 and L_2 are two complex nilpotent Lie subalgebras of $\mathscr{L}(E_1)^{\mathrm{op}}$ and $\mathscr{L}(E_2)^{\mathrm{op}}$, respectively, we consider the tensor product $E_1 \otimes E_2$ of E_1 and E_2, and the nilpotent Lie subalgebras $\tilde{L}_1$ and $\tilde{L}_2$ of $\mathscr{L}(E_1 \otimes E_2)^{\mathrm{op}}$, defined by

$$\tilde{L}_1 = \{x \otimes 1;\ x \in L_1\}, \qquad \tilde{L}_2 = \{1 \otimes y;\ y \in L_2\},$$

where 1 denotes the identity of the corresponding space. As in $\mathscr{L}(E_1 \otimes E_2)$ we have $[\tilde{L}_1, \tilde{L}_2] = 0$, the set

$$\{x \otimes 1 + 1 \otimes y;\ x \in L_1,\ y \in L_2\}$$

may be viewed as a direct product $L_1 \times L_2$ of the Lie algebras L_1 and L_2, with the natural bracket of $\mathscr{L}(E_1 \otimes E_2)^{\mathrm{op}}$. Furthermore, $\tilde{L}_1$ and $\tilde{L}_2$ are two ideals of the complex nilpotent Lie algebra $L_1 \times L_2$. Our objective is to compute the joint spectra of $L_1 \times L_2$ on $E_1 \otimes E_2$. The following theorem describes the situation.

THEOREM 8. *Let E_1 and E_2 be two complex finite dimensional vector spaces, and L_1 and L_2 two complex nilpotent subalgebras of $\mathscr{L}(E_1)^{\mathrm{op}}$ and $\mathscr{L}(E_2)^{\mathrm{op}}$, respectively. Then*

$$\mathrm{Sp}(L_1 \times L_2, E_1 \otimes E_2) = \mathrm{Sp}(L_1, E_1) \times \mathrm{Sp}(L_2, E_2),$$
$$\sigma_{\delta k}(L_1 \times L_2, E_1 \otimes E_2) = \mathrm{Sp}(L_1, E_1) \times \mathrm{Sp}(L_2, E_2),$$
$$\sigma_{\pi k}(L_1 \times L_2, E_1 \otimes E_2) = \mathrm{Sp}(L_1, E_1) \times \mathrm{Sp}(L_2, E_2),$$

where $0 \leqslant k \leqslant n + m$, $n = \dim L_1$, *and* $m = \dim L_2$.

Proof. Let us consider the decomposition of E_1 (respectively, E_2) as the direct sum of its weight L_1-submodules (respectively, its weight L_2-submodules):

$$E_1 = \bigoplus_{\alpha \in \Phi} E_{1\alpha}, \qquad E_2 = \bigoplus_{\beta \in \Psi} E_{2\beta},$$

where $\Phi = \{\alpha \in L_1^*;\ \alpha \text{ is a weight of } L_1 \text{ for } E_1\} = \mathrm{Sp}(L_1, E_1)$ and $\Psi = \{\beta \in L_2^*;\ \beta \text{ is a weight of } L_2 \text{ for } E_2\} = \mathrm{Sp}(L_2, E_2)$. Then

$$E_1 \otimes E_2 = \bigoplus_{(\alpha, \beta) \in \Phi \times \Psi} E_{1\alpha} \otimes E_{2\beta}.$$

An easy calculation shows that each $E_{1\alpha} \otimes E_{2\beta}$, $(\alpha, \beta) \in \Phi \times \Psi$, is an $L_1 \times L_2$-submodule of $E_1 \otimes E_2$. Besides, as $E_{1\alpha} \neq 0$ and $E_{2\beta} \neq 0$, we have $E_{1\alpha} \otimes E_{2\beta} \neq 0$. In addition, by the definition of Φ and Ψ, if $(\alpha_1, \beta_1) \neq (\alpha_2, \beta_2)$, there exists an element (x, y) of $L_1 \times L_2$ such that $(\alpha_1, \beta_1)(x, y) \neq (\alpha_2, \beta_2)(x, y)$. Finally, in order to conclude the proof, by Theorem 4 and [5, Chapter II, Theorem 7], it is enough to prove that $E_{1\alpha} \otimes E_{2\beta}$ is the weight module of $L_1 \times L_2$ for the weight (α, β). However, this fact may be verified by a slight modification of [5, Chapter III, 4]. ∎

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